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Aequationes Mathematicae

A new graceful labeling for pendant graphs

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Abstract. A graceful labeling of a graph G with q edges is an injective assignment of labels from $\{0, 1, \ldots, q\}$ to the vertices of G so that when each edge is assigned the absolute value of the difference of the vertex labels it connects, the resulting edge labels are distinct. A labeling of the first kind for coronas $C_n \odot K_1$ occurs when vertex labels 0 and $q = 2n$ are assigned to adjacent vertices of the n-gon. A labeling of the second kind occurs when $q = 2n$ is assigned to a pendant vertex. Previous research has shown that all coronas $C_n \odot K_1$ have a graceful labeling of the second kind. In this paper we show that all coronas $C_n \odot K_1$ with $n \equiv 3, 4 \pmod 8$ also have a graceful labeling of the first kind.

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1. Introduction

Let $G = (V(G), E(G))$ be a finite simple connected graph with vertex set $V(G)$ and edge set $E(G)$ where $e = uv$ if and only if edge e connects vertex u to vertex v. A function f is called a *graceful labeling* of a graph G with q edges if $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E(G) \to \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. This type of graph labeling, first introduced by Rosa in 1967 [5] as a *β-valuation*, was used as a tool for decomposing a complete graph into isomorphic subgraphs. Graceful labelings have since been applied in areas such as coding theory, radar, radio astronomy, and circuit design.

Many of the results about graph labelings, including graceful labelings, are collected and updated in a survey by Gallian [3]. The interested reader can consult this survey for more information about the subject.

One such result made by Hebbare [4] is that of the graphs C_n (commonly known as cycles): C_n is graceful if and only if

$$n \equiv 0 \text{ or } n \equiv 3 \pmod 4 \tag{1}$$

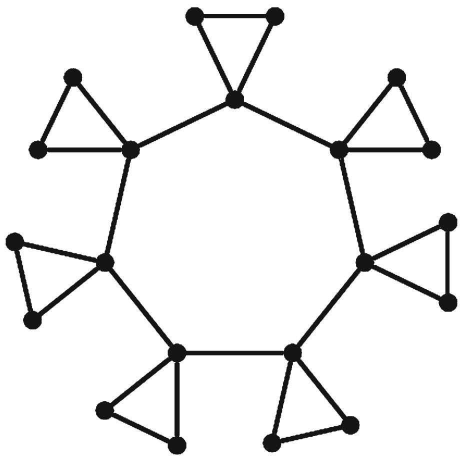
Since these graphs have the same number of vertices and edges, only one of the integers $0, 1, 2, \ldots, n$ is not used as a vertex label in the graceful labeling. It then follows that whenever (1) holds, a graceful labeling of C_n can be obtained by labeling the n vertices using the ordered sequence of labels

$$0, n, 1, n-1, 2, n-2, \ldots \tag{2}$$

where the integer $\lfloor \frac{n+1}{4} \rfloor$ is omitted.

The fact that C_n is a graceful graph only if (1) is satisfied led Frucht [1] to investigate a similar family of graphs, which he described as "polygons (=cycles) with pendant points attached; more precisely as the coronas $C_n \odot K_1$." The corona $G_1 \odot G_2$ of two graphs, as defined by Frucht and Harary [2], is the graph obtained by taking one copy of G_1, which has p_1 vertices, and p_1 copies of G_2, and then joining the ith vertex of G_1 by an edge to every vertex in the ith copy of G_2.

Example 1. Here is an image of the corona $C_7 \odot K_2$:



While investigating graceful labelings of coronas $C_n \odot K_1$ (which we will now refer to as pendant graphs), Frucht [1] observed that there are three possible kinds of graceful labelings for this family of graphs. This follows from the fact that the labels 0 and $2n$ must be adjacent in order for the induced edge labeling to be bijective. He described a graceful labeling of a pendant graph to be of the:

1. first kind if the labels 0 and $q = 2n$ are assigned to adjacent vertices of the n-gon.

2. second kind if $q = 2n$ is assigned to a pendant vertex.
3. third kind if 0 is assigned to a pendant vertex.

Frucht's proof that all pendant graphs are graceful only produces graceful labelings of the second kind. He then conjectured that a graceful labeling of the first kind exists for all pendant graphs. In this paper we will prove that if $n \equiv 3$ or $4 \pmod 8$, then pendant graphs have graceful labelings of the first kind.

2. Pendant graphs with $n \equiv 4 \pmod 8$

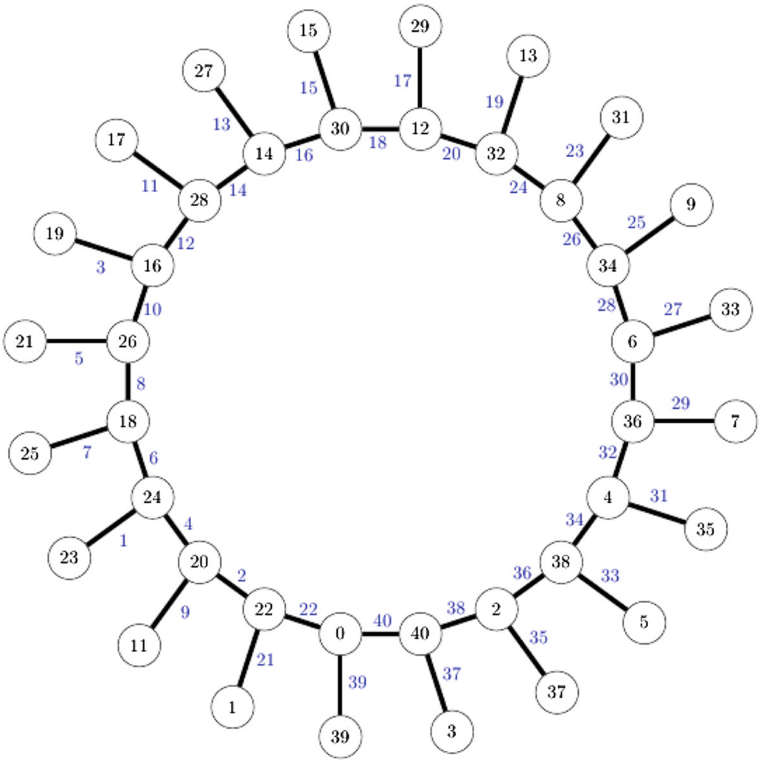
For the functions included in the remaining sections, the set of cycle vertices will be denoted $\{v_1, v_2, \ldots, v_n\}$ and the set of pendant vertices will be denoted $\{u_1, u_2, \ldots, u_n\}$, where v_i and u_j are adjacent if and only if $i = j$.

Theorem 1. *If* $n \equiv 4 \pmod 8, n > 12$, *the following function produces a graceful labeling for* $C_n \odot K_1$*:*

$$f(v_i) = \begin{cases} 0 & \textit{if} \quad i = n \\ 2n - i + 1 & \textit{if} \quad i = 1, 3, 5, \ldots, n-1 \\ i + 2 & \textit{if} \quad i = \frac{n}{2}, \frac{n}{2}+2, \frac{n}{2}+4, \ldots, n-2 \\ i & \textit{if} \quad i = 2, 4, 6, \ldots, \frac{n}{2}-2 \end{cases} \tag{3}$$

$$f(u_i) = \begin{cases} 2n - 1 & \textit{if} \quad i = n \\ 1 & \textit{if} \quad i = n-1 \\ \frac{n}{2}+1 & \textit{if} \quad i = n-2 \\ 2n - i + 1 & \textit{if} \quad i = \frac{3n}{4}+1, \frac{3n}{4}+3, \frac{3n}{4}+5, \ldots, n-4 \\ i + 6 & \textit{if} \quad i = \frac{3n}{4}, \frac{3n}{4}+2, \frac{3n}{4}+4, \ldots, n-3 \\ i + 5 & \textit{if} \quad i = \frac{3n}{4}-1 \\ i + 4 & \textit{if} \quad i = \frac{n}{2}-1, \frac{n}{2}+1, \frac{n}{2}+3, \ldots, \frac{3n}{4}-2 \\ 2n - i - 1 & \textit{if} \quad i = 2, 4, 6, \ldots, \frac{3n}{4}-3 \\ i + 2 & \textit{if} \quad i = 1, 3, 5, \ldots, \frac{n}{2}-3 \end{cases}$$

Example 2. Here is a graceful labeling of the first kind for a pendant graph with $n = 20$:



Before proving that the function from Theorem 1 produces a graceful labeling, some observations can be made from Example 2.

1 All cycle vertices have even labels. This induces all of the even edge labels from 2 to 40.
2 All pendant vertices have odd labels. This induces all of the odd edge labels from 1 to 39.
3 The labeling begins at the bottom of the example at the vertex labeled 40 and continues counterclockwise around the graph.

Proof. Through close examination of the above function f, it can be seen that the induced edge labeling is bijective. As f uses Hebbare's [4] graceful cycle labeling and multiplies each of these cycle vertex labels by 2, the resulting labeling uses all of the even edge labels from 2 to $2n$ as cycle edge labels. Thus, all of the pendant edges must have odd labels, which implies that all of the pendant vertices must have odd vertex labels.

By calculating the differences in the cycle and vertex labels, it can be seen that f induces all of the odd edge labels from $n+3$ to $2n-3$ when $1 \leq i \leq \frac{n}{2} - 2$. Specifically, when i is odd, the edge labels are $f(v_i) - f(u_i) =$

$(2n-i+1)-(i+2)=2n-2i-1$, which produces values that start at $2n-3$ and decrease by 4 as i increases by 2 until $i=\frac{n}{2}-3$ (which always produces the label $n+5$). When i is even, the edge labels are $(2n-i-1)-i=2n-2i-1$, which produces values that start at $2n-5$ and decrease by 4 as i increases by 2 until $i=\frac{n}{2}-2$ (which always produces the label $n+3$).

The labeling pattern changes at $i=\frac{n}{2}-1$, inducing the edge label $n-1$ instead of $n+1$, which is the second largest unused edge label (since $2n-1$ has yet to be induced). This pattern continues for $\frac{n}{2}-1\leq i\leq\frac{3n}{4}-2$, inducing the edge labels from $\frac{n}{2}+1$ to $n-1$. When i is odd, the edge labels are $(2n-i+1)-(i+4)=2n-2i-3$, which produces values that start at $n-1$ and decrease by 4 as i increases by 2 until $i=\frac{3n}{4}-2$ (which always produces the label $\frac{n}{2}+1$). When i is even, the edge labels are $(2n-i-1)-(i+2)=2n-2i-3$, which produces values that start at $n-3$ and decrease by 4 as i increases by 2 until $i=\frac{3n}{4}-3$ (which always produces the label $\frac{n}{2}+3$).

For $i=\frac{3n}{4}-1$, the induced edge is always 3 $[(i+5)-(i+2)=3]$. When $\frac{3n}{4}\leq i\leq n-3$, the induced edges are $2n-2i-5$ for odd values of i $[(2n-i+1)-(i+6)=2n-2i-5]$ and $2n-2i-1$ for even values of i $[(2n-i+1)-(i+2)=2n-2i-1]$. This induces the edges from 5 to $\frac{n}{2}-3$. Specifically, when $i=\frac{3n}{4}$, the edge label $\frac{n}{2}-5$ is produced. As i increases by 2, the edge labels decrease by 4 until $i=n-3$ (which always produces the label 1 since $2n-2(n-3)-5=1$). When $i=\frac{3n}{4}+1$, the edge label $\frac{n}{2}-3$ is produced. As i increases by 2, the edge labels decrease by 4 until $i=n-4$ (which always produces the label 7 since $2n-2(n-4)-1=7$).

At this point, all even edges have been accounted for and all the odd edges except $n+1$, $\frac{n}{2}-1$, and $2n-1$ have also been produced. When $i=n-2$, the function induces the edge label $\frac{n}{2}-1$ $[(i+2)-(\frac{n}{2}+1)=n-(\frac{n}{2}+1)=\frac{n}{2}-1]$. When $i=n-1$, the function induces the edge label $n+1$ $[(2n-i+1)-1=2n-(n-1)=1]$. Finally, when $i=n$, the edge label induced is $2n-1$ $[(2n-1)-0=2n-1]$. Thus, all edge values have been produced without any vertex or edge labels being repeated, so the pendant graph is gracefully labeled. □

Remark 1. Graceful labelings of the first kind exist for pendant graphs with values of n smaller than the restriction given in Theorem 1. For these small values of n, some vertices belong to multiple portions of the piecewise function. A graceful labeling can be produced using the function from Theorem 1 by letting the first label a vertex is assigned be the label of that vertex. Vertex labels are then assigned in the order they are listed in the piecewise function. For example, in the pendant graph $n=12$, the vertex u_{10} corresponds to both $i=n-2$ and $i=\frac{3n}{4}+1$. By following the rule stated above, this vertex is assigned the label $f(u_{10})=\frac{12}{2}+1=7$.

3. Pendant graphs with $n \equiv 3 \pmod 8$

This section contains two nearly identical functions for gracefully labeling pendant graphs with n-cycles where $n \equiv 3 \pmod{16}$ and $n \equiv 11 \pmod{16}$. When combined, these algorithms provide graceful labelings for all pendant graphs with n-cycles where $n \equiv 3 \pmod 8$.

Theorem 2. *If $n \equiv 3 \pmod{16}, n > 19$, the following function produces a graceful labeling for $C_n \odot K_1$:*

$$f(v_i) = \begin{cases} 0 & \textit{if} \quad i = n \\ 2n - i + 1 & \textit{if} \quad i = 1, 3, 5, \ldots, n-2 \\ i + 2 & \textit{if} \quad i = \frac{n+1}{2}, \frac{n+1}{2} + 2, \frac{n+1}{2} + 4, \ldots, n-1 \\ i & \textit{if} \quad i = 2, 4, 6, \ldots, \frac{n+1}{2} - 2 \end{cases} \tag{4}$$

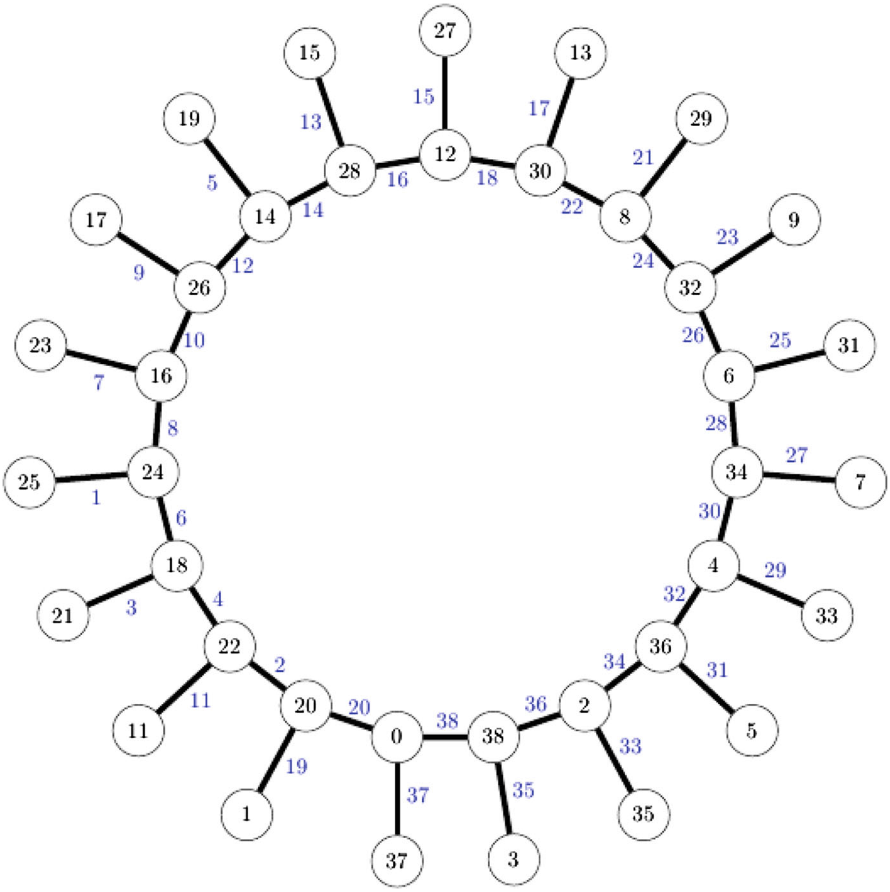
$$f(u_i) = \begin{cases} 2n - 1 & \textit{if} \quad i = n \\ 1 & \textit{if} \quad i = n-1 \\ \frac{n+1}{2} + 1 & \textit{if} \quad i = n-2 \\ n + 2 & \textit{if} \quad i = n-3 \\ 2n - i - 1 & \textit{if} \quad i = \frac{3n-1}{4} \\ 2n - i + 2 & \textit{if} \quad i = \frac{3n-1}{4} + 1 \\ 2n - i - 7 & \textit{if} \quad i = \frac{3n-1}{4} - 2, \frac{3n-1}{4} + 2, \frac{3n-1}{4} + 6, \ldots, n-7 \\ 2n - i + 1 & \textit{if} \quad i = \frac{3n-1}{4} + 4, \frac{3n-1}{4} + 8, \frac{3n-1}{4} + 12, \ldots, n-5 \\ i + 2 & \textit{if} \quad i = \frac{3n-1}{4} + 3, \frac{3n-1}{4} + 5, \frac{3n-1}{4} + 7, \ldots, n-4 \\ 2n - i - 1 & \textit{if} \quad i = 2, 4, 6, \ldots, \frac{3n-1}{4} - 4 \\ i + 4 & \textit{if} \quad i = \frac{n+1}{2} - 1, \frac{n+1}{2} + 1, \frac{n+1}{2} + 3, \ldots, \frac{3n-1}{4} - 1 \\ i + 2 & \textit{if} \quad i = 1, 3, 5, \ldots, \frac{n+1}{2} - 3 \end{cases}$$

Theorem 3. *If $n \equiv 11 \pmod{16}, n > 27$, the following function produces a graceful labeling for $C_n \odot K_1$:*

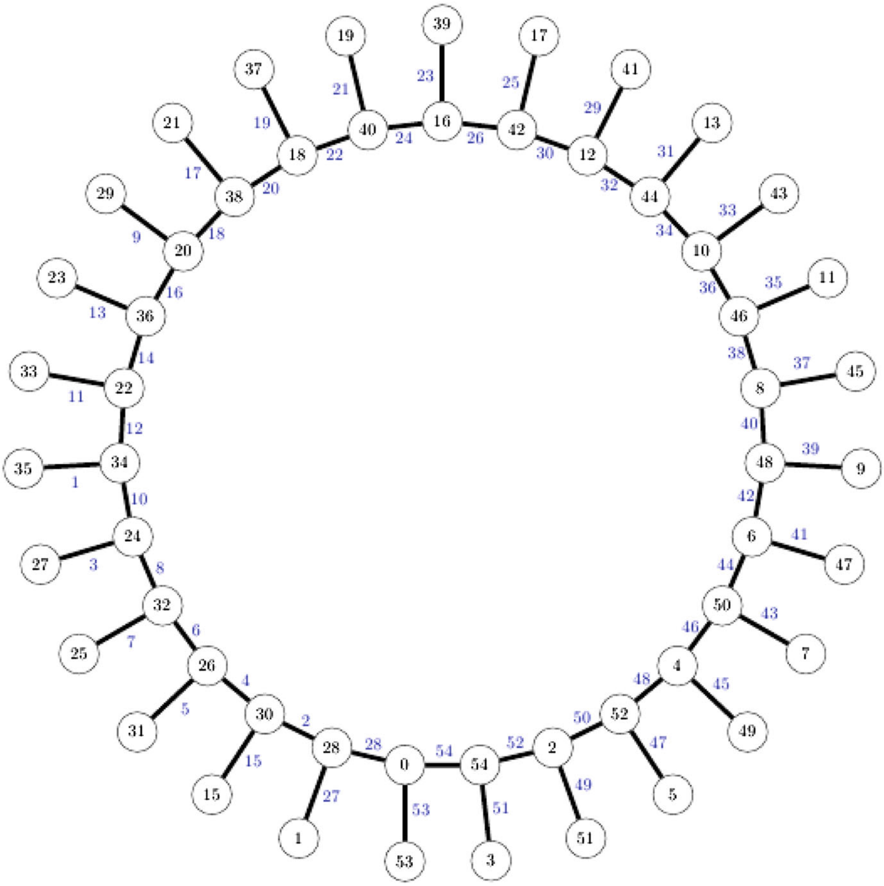
$$f(v_i) = \begin{cases} 0 & \textit{if} \quad i = n \\ 2n - i + 1 & \textit{if} \quad i = 1, 3, 5, \ldots, n-2 \\ i + 2 & \textit{if} \quad i = \frac{n+1}{2}, \frac{n+1}{2} + 2, \frac{n+1}{2} + 4, \ldots, n-1 \\ i & \textit{if} \quad i = 2, 4, 6, \ldots, \frac{n+1}{2} - 2 \end{cases} \tag{5}$$

$$f(u_i) = \begin{cases} 2n-1 & \text{if} \quad i = n \\ 1 & \text{if} \quad i = n-1 \\ \frac{n+1}{2}+1 & \text{if} \quad i = n-2 \\ n+4 & \text{if} \quad i = n-3 \\ n & \text{if} \quad i = n-5 \\ 2n-i-1 & \text{if} \quad i = \frac{3n-1}{4} \\ 2n-i+2 & \text{if} \quad i = \frac{3n-1}{4}+1 \\ 2n-i-7 & \text{if} \quad i = \frac{3n-1}{4}-2, \frac{3n-1}{4}+2, \frac{3n-1}{4}+6, \ldots, n-9 \\ 2n-i+1 & \text{if} \quad i = \frac{3n-1}{4}+4, \frac{3n-1}{4}+8, \frac{3n-1}{4}+12, \ldots, n-7 \\ i+2 & \text{if} \quad i = \frac{3n-1}{4}+3, \frac{3n-1}{4}+5, \frac{3n-1}{4}+7, \ldots, n-4 \\ 2n-i-1 & \text{if} \quad i = 2, 4, 6, \ldots, \frac{3n-1}{4}-4 \\ i+4 & \text{if} \quad i = \frac{n+1}{2}-1, \frac{n+1}{2}+1, \frac{n+1}{2}+3, \ldots, \frac{3n-1}{4}-1 \\ i+2 & \text{if} \quad i = 1, 3, 5, \ldots, \frac{n+1}{2}-3 \end{cases}$$

Example 3. Here is a graceful labeling of the first kind for a pendant graph with $n = 19$:



Example 4. Here is a graceful labeling of the first kind for a pendant graph with $n = 27$:



Proof (*of Theorems 2 and 3*). Similarly to the proof of (3), the cycle vertex labels of (4) and (5) use Hebbare's [4] graceful cycle labeling and multiplies each of these cycle vertex labels by 2. This resulting labeling uses all of the even edge labels from 2 to $2n$ as the cycle edge labels. As functions (4) and (5) have identical vertex labelings until $i = \frac{3n+1}{4} - 2$, the proof will address both labelings together up to this value i from where both functions will be considered separately.

By calculating the differences in the cycle and vertex labels, it can be seen that the above functions induce all of the odd edge labels from $n+2$ to $2n-3$ when $1 \le i \le \frac{n+1}{2} - 2$. When i is odd, the edge labels are $f(v_i) - f(u_i) = (2n - i + 1) - (i + 2) = 2n - 2i - 1$, which produces values that start at $2n - 3$

and decrease by 4 as i increases by 2 until $i = \frac{n+1}{2} - 3$ (which always produces the label $n+4$). When i is even, the edge labels are $(2n-i-1)-i = 2n-2i-1$, which produces values that start at $2n-5$ and decrease by 4 as i increases by 2 until $i = \frac{n+1}{2} - 2$ (which always produces the label $n+2$).

The functions then change at $i = \frac{n+1}{2} - 1$, inducing the edge label $n-2$ instead of n, which is the second largest unused edge label (since $2n-1$ has yet to be induced). The previous pattern then continues for $\frac{n+1}{2} - 1 \le i \le \frac{3n-1}{4} - 4$, inducing the edge labels from $\frac{n+1}{2} + 3$ to $n-2$. When i is odd, the edge labels are $(2n-i+1)-(i+4) = 2n-2i-3$, which produces values that start at $n-2$ and decrease by 4 as i increases by 2 until $i = \frac{3n-1}{4} - 4$ (which always produces the label $\frac{n+1}{2} + 5$). When i is even, the edge labels are $(2n-i-1)-(i+2) = 2n-2i-3$, which produces values that start at $n-4$ and decrease by 4 as i increases by 2 until $i = \frac{3n-1}{4} - 3$ (which always produces the label $\frac{n+1}{2} + 3$).

When $i = \frac{3n-1}{4} - 2$, the induced edge is $\frac{n+1}{2} - 5$ $[(2n-i-7)-(i+2) = 2n-2i-9 = \frac{4n-3n+1}{2} - 5 = \frac{n+1}{2} - 5]$. For $i = \frac{3n-1}{4} - 1$, the induced edge is always $\frac{n+1}{2} - 1$ $[(2n-i+1)-(i+4) = 2n-2i-3 = \frac{4n-3n+1}{2} - 1 = \frac{n+1}{2} - 1]$. When $\frac{3n-1}{4}$, the induced edge is $\frac{n+1}{2} - 3$ $[(2n-i-1)-(i+2) = 2n-2i-3 = \frac{4n-3n+1}{2} - 3 = \frac{n-1}{2} - 3]$. For $i = \frac{3n-1}{4} + 1$, the induced edge is always 1 $[(2n-i+2)-(2n-i+1) = 1]$.

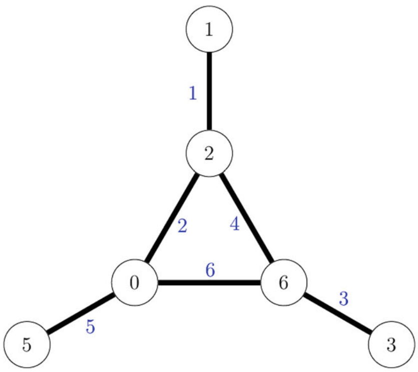
Now we will consider function (4) and function (5) separately. First, we will focus on (4). For $\frac{3n-1}{4} + 2 \le i \le n-4$, a new pattern emerges. The induced edges are $2n-2i-1$ for odd values of i $[(2n-i+1)-(i+2) = 2n-2i-1]$, $2n-2i-9$ for even values of i where $i \equiv 0 \pmod 4$ $[(2n-i-7)-(i+2) = 2n-2i-9]$, and $2n-2i-1$ for even values of i where $i \equiv 2 \pmod 4$ $[(2n-i+1)-(i+2) = 2n-2i-1]$. This induces the edges from 5 to $\frac{n+1}{2} - 7$. When $i = \frac{3n-1}{4} + 2$, the edge label $\frac{n+1}{2} - 13$ is produced. As i increases by 4, the edge labels decrease by 8 until $i = n-7$ (which always produces the label 5 since $2n-2(n-7)-9 = 5$). When $i = \frac{3n-1}{4} + 3$, the edge label $\frac{n+1}{2} - 7$ is produced. As i increases by 2, the edge labels decrease by 4 until $i = n-4$ (which always produces the label 7 since $2n-2(n-4)-1 = 7$). When $i = \frac{3n-1}{4} + 4$, the edge label $\frac{n+1}{2} - 9$ is produced. As i increases by 4, the edge labels decrease by 8 until $i = n-5$ (which always produces the label 9 since $2n-2(n-5)-1 = 9$).

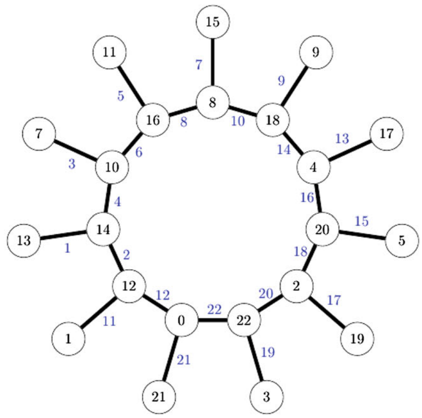
At this point, all even edges have been accounted for and all the odd edges except n, $\frac{n+1}{2} + 1$, 3, and $2n-1$ have also been produced. For $i = n-3$, the induced edge label is always 3 $[(n+2)-(n-3+2) = 3]$. When $i = n-2$, the function induces the edge label $\frac{n+1}{2} + 1$ $[(2n-(n-2)+1)-(\frac{n+1}{2} + 1) = n+3-(\frac{n-1}{2} + 2) = \frac{2n-n+1}{2} + 1 = \frac{n+1}{2} + 1]$. When $i = n-1$, the function induces the edge label n $[(i)-1 = (n-1+2)-1 = n]$. Finally, when $i = n$, the edge label induced is $2n-1$ $[(2n-1)-0 = 2n-1]$. Thus, all edge values have been produced without any vertex or edge labels being repeated, so the pendant graph is gracefully labeled.

Similarly for (5), a new pattern emerges for $\frac{3n-1}{4}+2 \le i \le n-6$. The induced edges are $2n-2i-1$ for odd values of i $[(2n-i+1)-(i+2)=2n-2i-1]$, $2n-2i-1$ for even values of i where $i \equiv 2 \pmod 4$ $[(2n-i+1)-(i+2)=2n-2i-1]$, and $2n-2i-9$ for even values of i where $i \equiv 0 \pmod 4$ $[(2n-i-7)-(i+2)=2n-2i-9]$. This induces the edges from 11 to $\frac{n+1}{2}-7$. When $i=\frac{3n-1}{4}+2$, the edge label $\frac{n+1}{2}-13$ is produced. As i increases by 4, the edge labels decrease by 8 until $i=n-9$ (which always produces the label 9 since $2n-2(n-9)-9=9$). When $i=\frac{3n-1}{4}+3$, the edge label $\frac{n+1}{2}-7$ is produced. As i increases by 2, the edge labels decrease by 4 until $i=n-6$ (which always produces the label 11 since $2n-2(n-6)-1=11$). When $i=\frac{3n-1}{4}+4$, the edge label $\frac{n+1}{2}-9$ is produced. As i increases by 4, the edge labels decrease by 8 until $i=n-7$ (which always produces the label 13 since $2n-2(n-7)-1=13$).

At this point, all even edges have been accounted for and all the odd edges except n, $\frac{n+1}{2}+1$, 3, 5, 7, and $2n-1$ have also been produced. For $i=n-5$, the induced edge label is always 3 $[n-(n-5+2)=3]$. When $i=n-4$, the function induces the edge label 7 $[(2n-(n-4)+1)-(n-4+2)=7]$. For $i=n-3$, the induced edge label is always 5 $[(n+4)-(n-3+2)=5]$. When $i=n-2$, the function induces the edge label $\frac{n+1}{2}+1$ $[(2n-(n-2)+1)-(\frac{n+1}{2}+1)= n+3-(\frac{n-1}{2}+2)=\frac{2n-n+1}{2}+1=\frac{n+1}{2}+1]$. When $i=n-1$, the function induces the edge label n $[(i)-1=(n-1+2)-1=n]$. Finally, when $i=n$, the edge label induced is $2n-1$ $[(2n-1)-0=2n-1]$. Thus, all edge values have been produced without any vertex or edge labels being repeated, so the pendant graph is gracefully labeled. □

Remark 2. Interestingly, a graceful labeling of the first kind following either (4) or (5) for the pendant graphs with cycles $n=3$ and $n=11$ have not been found. However, the following images are examples of a graceful labeling of the first kind for these pendant graphs.





The above theorems have shown that graceful labelings of the first kind exist for pendant graphs with n-cycles where $n \equiv 3, 4 \pmod 8$. The author's future work will include investigating the remaining congruence classes for which no algorithms have yet been found.

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